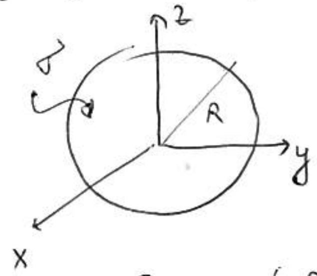
Диракова δ-фја

$$\rho(\vec{r}) = \sum_\alpha q_\alpha \delta^{(3)}(\vec{r}-\vec{r}_\alpha) = \int_\Sigma \sigma\, ds'\, \delta^{(3)}(\vec{r}-\vec{r}\,') = \int_C \lambda(\vec{r}\,')\, dl'\, \delta^{(3)}(\vec{r}-\vec{r}\,')$$

1. Користећи се особинама δ-фје одредити запреминске густине наелектрисања у Декартовим, сферним и цилиндричним координатама за следеће конфигурације:

(i) сферна површ, полупречника R, равномерно наелектрисана густином наелектрисања σ.



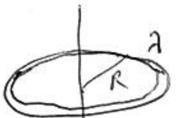
сферне: $$\rho_S = \sum q_\alpha \delta(\vec{r}-\vec{r}_\alpha) = \int \sigma \cdot \cancel{R^2 \sin\theta'}\, \delta(r-R)\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')\,\frac{1}{\cancel{r^2\sin\theta}}\,d\theta' d\varphi'$$

$$\boxed{\rho_S = \sigma\delta(r-R)}$$

$$\rho_C = \sigma\left(\delta(r-R)+\delta(r+R)\right) = 2R\sigma\delta(r^2-R^2) = \boxed{2R\sigma\,\delta(r_c^2+z^2-R^2)}$$

$$\boxed{\rho_D = 2R\sigma\,\delta(x^2+y^2+z^2-R^2)}$$

(ii) танак прстен, полупречника R, равномерно наелектрисан густином наелектр. λ.

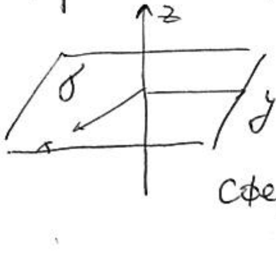


цилиндричне: $$\rho_C = \int \lambda \underbrace{R\,d\varphi'}_{dl'} \frac{1}{\cancel{r}}\,\delta(r_c-R)\,\delta(z)\,\delta(\varphi-\varphi') = \boxed{\lambda\delta(r_c-R)\,\delta(z)}$$

сферне $$\rho_S = \int \lambda R\, d\varphi' \underbrace{\frac{1}{r^2\sin\theta}}_{=1}\,\delta(r-R)\,\delta\!\left(\theta-\tfrac{\pi}{2}\right)\delta(\varphi-\varphi') = \boxed{\frac{\lambda}{R}\delta(r-R)\,\delta\!\left(\theta-\tfrac{\pi}{2}\right)}$$

Декартове $$\rho_D = \lambda\left(\delta(r_c-R)+\delta(r_c+R)\right)\delta(z) = \boxed{2\lambda R\,\delta(x^2+y^2-R^2)\,\delta(z)}$$

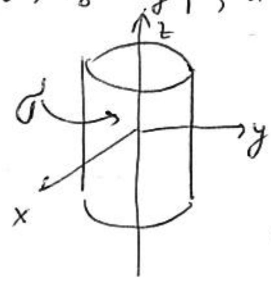
(iii) раван наелектрисана равномерно густином наелектрисања σ



Декартове: $$\rho_D = \int \sigma\, dx' dy'\, \delta(x-x')\,\delta(y-y')\,\delta(z-0) = \boxed{\sigma\delta(z)} = \rho_C$$
цилиндричне

сферне $$\rho_S = \int \sigma\, \cancel{r'} dr' d\varphi' \underbrace{\frac{1}{r^{\cancel{2}}\sin\theta}}_{=1}\,\delta(r-r')\,\delta\!\left(\theta-\tfrac{\pi}{2}\right)\delta(\varphi-\varphi') = \boxed{\frac{\sigma}{r}\delta\!\left(\theta-\tfrac{\pi}{2}\right)}$$

(iv) цилиндар, полупречника R, равномерно наелектрисан густином наелектрисања σ

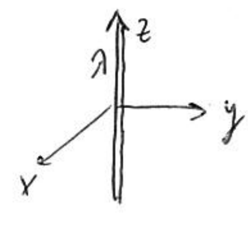


цилиндричне: $$\rho_C = \int \sigma \cancel{R}\, d\varphi' dz' \frac{1}{\cancel{r_c}}\,\delta(r_c-R)\,\delta(\varphi-\varphi')\,\delta(z-z') = \boxed{\sigma\delta(r_c-R)}$$

Декартове: $$\rho_D = \sigma\left(\delta(r_c-R)+\delta(r_c+R)\right) = \boxed{2R\sigma\,\delta(x^2+y^2-R^2)}$$

сферне $$\rho_S = \boxed{\sigma\delta(r\sin\theta-R)}$$

(v) бесконачна нит, равномерно наелектрисана густином наелектрисања λ



Декартове: $$\rho_D = \int \lambda\, dz'\, \delta(x-\overset{0}{x'})\,\delta(y-\overset{0}{y'})\,\delta(z-z') = \boxed{\lambda\delta(x)\,\delta(y)}$$

цилиндричне: φ - није дефинисано на z-оси па радимо за цилиндар полупречника R и наелектрисања $\sigma \cdot 2R\pi \cdot dz = \lambda dz \rightarrow \boxed{\sigma = \frac{\lambda}{2R\pi}}$ на крају $\lim\limits_{R\to 0}$

$$\rho_c = \lim_{R\to 0} \frac{\lambda}{2R\pi}\delta(r_c - R) = \lim_{R\to 0}\frac{\lambda}{2r_c\pi}\delta(r_c - R) = \boxed{\frac{\lambda}{2r_c\pi}\delta(r_c)}$$

сферне: $\rho_s = \dfrac{\lambda}{2r\sin\theta\,\pi}\delta(r\sin\theta) \rightarrow \theta = 0$ или $\theta = \pi$

Пробамо да упростимо

1° покушај $\rho_s = f(r)\,\delta(\sin\theta)$

интегралимо ρ_s у сферној љусци $dq = 2\lambda dr = \int_0^{\pi}\int_0^{2\pi} f(r)\,\delta(\sin\theta)\, r^2\sin\theta\, dr\, d\theta\, d\varphi = 0$

$\rightarrow$ Не ваља претпоставка!

2° покушај $\rho_s = f(r)\,\delta(1-\cos^2\theta) \rightarrow \theta = 0$ или $\theta = \pi$

опет интегралимо у сферној љусци $dq = 2\lambda dr = \int_0^{\pi}\int_0^{2\pi} f(r)\,\delta(1-\cos^2\theta)\, r^2\sin\theta\, dr\, d\theta\, d\varphi$

$$2\lambda dr = f(r) r^2 dr\cdot 2\pi \int_{-1}^{1}\delta(1-t^2)\,dt = f(r)\cdot r^2 dr\cdot 2\pi\int_{-1}^{1}\frac{1}{2}\Big[\underbrace{\delta(1-t)}_{1/2} + \underbrace{\delta(1+t)}_{1/2}\Big]dt$$

$t = \cos\theta$; $dt = -\sin\theta\, d\theta$

$$2\lambda dr = f(r)\cdot r^2 dr\cdot 2\pi\,\frac{1}{2} = f(r)\cdot r^2\pi\, dr \rightarrow \boxed{f(r) = \frac{2\lambda}{r^2\pi}}$$

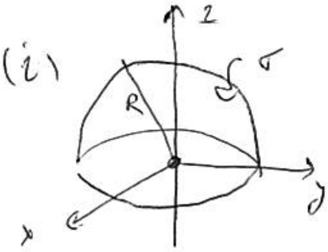
$$\boxed{\rho_s = \frac{2\lambda}{r^2\pi}\delta(1-\cos^2\theta)}$$

2. Користећи се особинама δ и η ф-ја, одредити запреминску густину наелектрисања у Декартовим, сферним и цилиндричним координатама за следеће конфигурације:

(i) полусфера полупречника R, равномерно наелектрисана густином σ

(ii) танак диск, полупречника R, равномерно наелектрисан густином σ.

(i)



$$\rho = \int \sigma(\vec r\,')ds'\,\delta(\vec r - \vec r\,')$$

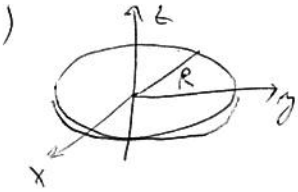
$$\rho_S = \int_0^{\pi/2}\int_0^{2\pi}\sigma\cdot \cancel{R^2\sin\theta'}\,\frac{d\theta' d\varphi'}{\cancel{r^2\sin\theta}}\,\delta(r-R)\,\delta(\theta-\theta')\,\delta(\varphi-\varphi') = \sigma\delta(r-R)\int_0^{\pi/2}\delta(\theta-\theta')d\theta'$$

$$= \sigma\delta(r-R)\int_0^{\pi}\eta\left(\frac{\pi}{2}-\theta'\right)\delta(\theta-\theta')\,d\theta' = \boxed{\sigma\delta(r-R)\,\eta\left(\frac{\pi}{2}-\theta\right)}$$

Декартове: $\rho_D = \sigma(\delta(r-R) + \delta(r+R))\,\eta\left(\frac{\pi}{2}-\theta\right) = \boxed{2\sigma R\,\delta(x^2+y^2+z^2-R^2)\,\delta(z)}$ $\left(\theta < \frac{\pi}{2} \Leftrightarrow z > 0\right)$

цилиндричне $\rho_C = \boxed{2\sigma R\,\delta(r_c^2 + z^2 - R^2)\,\delta(z)}$

(ii)



$$\rho(\vec r) = \int_{R\in S} \sigma(\vec r\,')\, dS'\, \delta^{(3)}(\vec r - \vec r\,')$$

$$= \int_0^R\int_0^{2\pi} \sigma\, \cancel{r_c'}\, dr_c'\, d\varphi'\, \frac{\delta(r_c - r_c')\,\delta(\varphi-\varphi')\,\delta(z)}{\cancel{r_c}} = \int_0^R \sigma\,\delta(r_c - r_c')\,\delta(z)\, dr_c'$$

$$= \int_0^\infty \sigma\,\delta(r_c - r_c')\,\delta(z)\, dr_c'\, \eta(R - r_c') = \int_0^\infty \sigma\,\delta(r_c - r_c')\,\delta(z)\,\eta(R-r_c)\, dr_c' =$$

$$= \boxed{\sigma\,\delta(z)\,\eta(R - r_c)}$$ цилиндричне

Декартове: $\rho(\vec r) = \sigma\delta(z)\,\eta(R^2 - x^2 - y^2)$

Сферне: $\rho(\vec r) = \sigma\,\delta(r\cos\theta)\,\eta(R - r\sin\theta) = \sigma\,\frac{1}{r}\,\delta(\theta - \frac{\pi}{2})\,\eta(R - r\sin\theta)$

$$= \boxed{\frac{\sigma}{r}\,\delta(\theta - \tfrac{\pi}{2})\,\eta(R - r)}$$

3. Одредити облик наелектрисаног тела и тип расподеле наелектрисања на њему ако је густина наелектрисања:

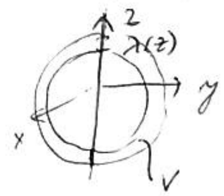
(i) $\rho = \frac{a}{r^2}\,\delta(1 - \cos^2\theta)$ (ii) $\rho = 2a\sigma\,\delta(x^2 - a^2)$ (iii) $\rho = 2aq\,\delta(x^2 - a^2)\,\delta(z)$

(iv) $\rho = 2q\sqrt{a^2+b^2}\,\delta(x^2 - 2ax - 2by + y^2)\,\delta(z)$ (v) $\rho = \frac{Q}{\pi ab}\,\delta(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1)\,\delta(z)$

(vi) $\rho = \frac{Q}{\pi abc}\,\delta(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1)$

(i) $\rho = \frac{a}{r^2}\,\delta(1 - \cos^2\theta)$ положај наелектрисања: $1 - \cos^2\theta = 0 \Rightarrow \cos\theta = \pm 1$

$\theta = 0$ или $\theta = \pi$ | z-оса |!



$\lambda(z) = \lambda(-z)$ из симетрије!

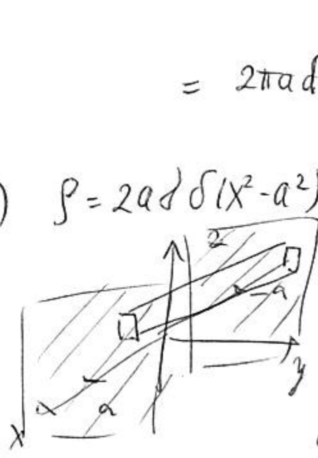
$\int_V \rho\, dV = 2\lambda(z)\,dz = dq$ наелектрисање у сферној љусци!

$$2\lambda(z)\,dz = \int_z^{z+dz}\int_0^{\pi}\int_0^{2\pi} \frac{a}{\cancel{r^2}}\,\delta(1-\cos^2\theta)\,\cancel{r^2}\sin\theta\, dr\, d\theta\, d\varphi$$

$$= a\cdot 2\pi\, dz \int \delta(1 - \cos^2\theta)\sin\theta\, d\theta = a\,2\pi\, dz \int_{-1}^{1} \delta(1 - t^2)\, dt$$

$$= 2\pi a\, dz \int_{-1}^{1} \frac{1}{2}\Big(\underbrace{\delta(1-t)}_{1/2} + \underbrace{\delta(1+t)}_{1/2}\Big)\, dt = \pi a\, dz \Rightarrow \boxed{\lambda(z) = \frac{\pi a}{2}} = \text{const}.$$

(ii) $\rho = 2a\sigma\,\delta(x^2 - a^2)$ положај наелектрисања $x^2 = a^2 \Rightarrow x = \pm a$ равни



$\sigma(y,z)$ исто за обе равни!

тражимо наелектрисање у бескон. тунелу квадратног попречног пресека

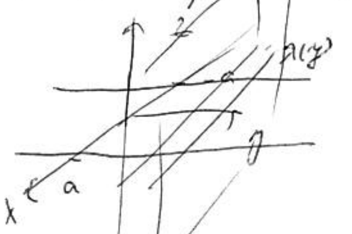
$$dq = \int \rho\, dV = 2\sigma(y,z)\, dy\, dz$$

$$2\sigma(y,z)\,dy\,dz = \int_{-\infty}^{+\infty}\int_y^{y+dy}\int_z^{z+dz} 2a\sigma\,\delta(x^2 - a^2)\, dx\, dy\, dz = 2a\sigma\, dy\, dz \int_{-\infty}^{+\infty} \frac{1}{2a}\underbrace{\Big(\delta(x-a) + \delta(x+a)\Big)}_{2} dx$$

$$= 2\sigma\, dy\, dz \Rightarrow \boxed{\sigma(y,z) = \sigma = \text{const}}$$

(iii) положај наелектрисања: $\rho = 2aq\,\delta(x^2-a^2)\,\delta(z)$ $\quad z=0,\ x=\pm a$

две праве у $z=0$ равни



$\lambda(y)$ исти на обе праве

количина наел. у простору између две равни $y=y_0$ и $y=y_0+dy$

$$dq = 2\lambda(y_0)\,dy = \int_{-\infty}^{+\infty}\int_{y_0}^{y_0+dy}\int_{-\infty}^{+\infty} 2aq\,\delta(x^2-a^2)\,\underline{\delta(z)}\,dx\,dy\,\underline{dz}$$

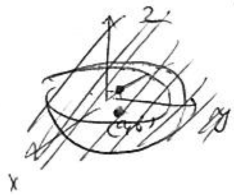
$$= 2aq\,dy\int_{-\infty}^{+\infty}\delta(x^2-a^2)\,dx = 2aq\,dy\int_{-\infty}^{+\infty}\frac{1}{2a}\underbrace{\big(\delta(x-a)+\delta(x+a)\big)dx}_{2}$$

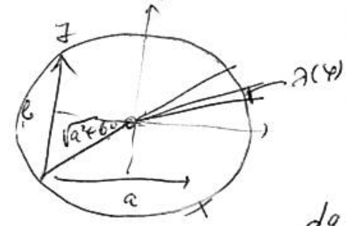
$$= 2q\,dy \quad \Rightarrow \quad \boxed{\lambda(y) = q = \text{const.}}$$

(iv) $\rho = 2q\sqrt{a^2+b^2}\,\delta(x^2-2ax-2by+y^2)\,\delta(z)$

положај наелектрисања: $x^2-2ax-2by+y^2=0$

$(x-a)^2+(y-b)^2 = a^2+b^2$ круг радијуса $R=\sqrt{a^2+b^2}$





смена: $x = a + r\cos\varphi$ $\quad \boxed{r=R}$ за круг

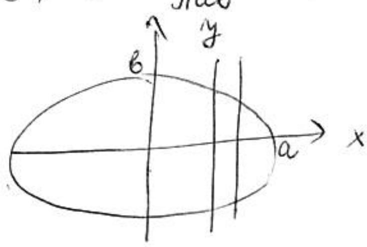
$y = b + r\sin\varphi$

$$dq = \lambda(\varphi)\,d\varphi\cdot\sqrt{a^2+b^2} = \int_0^{\infty}\int_{-\infty}^{+\infty}\int_{\varphi}^{\varphi+d\varphi}\rho\, r_c\,dr_c\,dz\,d\varphi$$

$$= \int_0^{\infty}\int_{-\infty}^{+\infty}\int_{\varphi}^{\varphi+d\varphi} 2q\sqrt{a^2+b^2}\,\delta(r_c^2-a^2-b^2)\,\delta(z)\,r_c\,dr_c\,\underline{dz}\,d\varphi = 2q\sqrt{a^2+b^2}\,d\varphi\int_0 \delta(r_c^2-a^2-b^2)\underbrace{r_c\,dr_c}_{\frac{1}{2}d(r_c^2-a^2-b^2)}$$

$$= q\sqrt{a^2+b^2}\,d\varphi \quad \Rightarrow \quad \boxed{\lambda(\varphi) = q}$$

(v) $\rho = \dfrac{Q}{\pi ab}\,\delta\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1\right)\delta(z)$ $\quad \boxed{\text{за } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1}$ елипса у $z=0$ равни



$\lambda(x) = \lambda(-x)$

$$y = \pm b\sqrt{1-\frac{x^2}{a^2}} \qquad dy = \pm b\,\frac{-\frac{x}{a^2}dx}{\sqrt{1-\frac{x^2}{a^2}}} = \mp\frac{bx\,dx}{a^2\sqrt{1-\frac{x^2}{a^2}}}$$

$$dl = \sqrt{dx^2+dy^2} = \left(dx^2\left(1+\frac{b^2x^2}{a^4-x^2a^2}\right)\right)^{1/2}$$

$$dl = dx\sqrt{\frac{a^4+(b^2-a^2)x^2}{a^2(a^2-x^2)}}$$

између x и $x+dx$ простор међу равнима

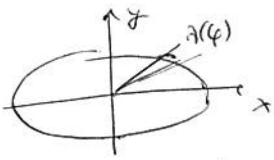
$$2\lambda(x)\,dl = \int_x^{x+dx}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho\,dV = \int_x^{x+dx}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{Q}{\pi ab}\,\delta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\underline{\delta(z)}\,dx\,dy\,dz$$

$$= \frac{Q}{\pi ab}\,dx\int_{-\infty}^{+\infty}\delta\left(\frac{y^2}{b^2}+\frac{x^2}{a^2}-1\right)dy = \frac{Q}{\pi ab}\,dx\int_{-\infty}^{+\infty}\frac{1}{2\sqrt{1-\frac{x^2}{a^2}}}\left(\delta\left(\frac{y}{b}-\sqrt{1-\frac{x^2}{a^2}}\right)+\delta\left(\frac{y}{b}+\sqrt{1-\frac{x^2}{a^2}}\right)\right)$$

$$= \frac{Q}{\pi a\cancel{b}}\,dx\,\frac{\cancel{b}}{\cancel{2}\sqrt{1-\frac{x^2}{a^2}}}\cdot\cancel{2} = \frac{Q\,dx}{\pi\sqrt{a^2-x^2}} = 2\lambda\,dx\sqrt{\frac{a^4+(b^2-a^2)x^2}{a^2(a^2-x^2)}}$$

$$\lambda = \frac{Q}{2\pi \cancel{\sqrt{a^2-x^2}}} \frac{a\cancel{\sqrt{a^2-x^2}}}{\sqrt{a^4+(b^2-a^2)x^2}} = \boxed{\frac{Qa}{2\pi\sqrt{a^4+(b^2-a^2)x^2}}}$$

2. начин: смена! $x = at\cos\varphi$; $y = bt\sin\varphi$ $\quad dV = abt\,dt\,d\varphi\,dz \quad t^2\cos^2\varphi \quad t^2\sin^2\varphi$



$$dq = \lambda(\varphi)\,dl = \int_0^{\infty}\int_{\varphi}^{\varphi+d\varphi}\int_{-\infty}^{+\infty} \rho\,dV = \int_0^{\infty}\int_{\varphi}^{\varphi+d\varphi}\int_{-\infty}^{+\infty} \frac{Q}{\pi ab}\,\delta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\delta(z)\,abt\,dt\,d\varphi\,dz$$

$$= \int_0^{\infty}\int_{\varphi}^{\varphi+d\varphi}\int_{-\infty}^{+\infty} \frac{Q}{\pi ab}\,\delta(t^2-1)\,\delta(z)\,abt\,dt\,d\varphi\,dz = \frac{Q}{\pi \cancel{ab}}\,d\varphi\,\cancel{ab}\int_0^{\infty} t\,\frac{1}{2}\left(\delta(t-1)+\underbrace{\cancel{\delta(t+1)}}_{0}\right)dt$$

$$= \frac{Q}{2\pi}\,d\varphi$$

$$dl^2 = dx^2+dy^2 = (-a\sin\varphi\,d\varphi)^2+(b\cos\varphi\,d\varphi)^2 = a^2\sin^2\varphi\,d\varphi^2+b^2\cos^2\varphi\,d\varphi^2$$

$t=1$ на елипси : $x = a\cos\varphi$, $y = b\sin\varphi$;

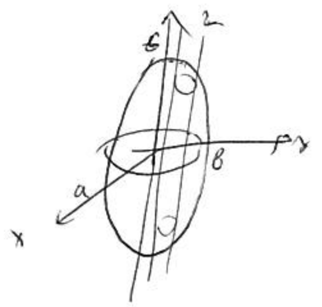
$$= (a^2\sin^2\varphi+b^2\cos^2\varphi)\,d\varphi^2$$

$$\lambda(\varphi)\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}\,\cancel{d\varphi} = \frac{Q}{2\pi}\cancel{d\varphi} \Rightarrow \lambda(\varphi) = \frac{Q}{2\pi\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}}$$

$$\lambda(x) = \frac{Q}{2\pi\sqrt{a^2\left(\frac{y}{b}\right)^2+b^2\left(\frac{x}{a}\right)^2}} = \frac{Q}{2\pi\sqrt{a^2\left(1-\frac{x^2}{a^2}\right)+b^2\frac{x^2}{a^2}}} = \boxed{\frac{Qa}{2\pi\sqrt{a^4+x^2(b^2-a^2)}}}$$

(vi) $\rho = \frac{Q}{2\pi abc}\,\delta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ елипсоид



$\sigma(x,y)$ има за $z>0$ и $z<0$

$$2\sigma(x,y)\,dS(x,y) = \int \rho\,dV$$

$$= \int_x^{x+dx}\int_y^{y+dy}\int_{-\infty}^{+\infty} \frac{Q}{2\pi abc}\,\delta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)dx\,dy\,dz$$

$dS = \sqrt{EG-F^2}\,du\,dv$; $E = \left(\frac{\partial\vec r}{\partial u}\right)^2$; $G = \left(\frac{\partial\vec r}{\partial v}\right)^2$; $F = \left(\frac{\partial\vec r}{\partial u}\right)\left(\frac{\partial\vec r}{\partial v}\right)$ Гаусови коефициенти

$\frac{\partial\vec r}{\partial x} = (1,0,z_x)$ $\quad \frac{\partial\vec r}{\partial y} = (0,1,z_y)$

$$z = \pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \qquad E = 1+z_x^2 = 1+\left(\frac{c\,\frac{x}{a^2}}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\right)^2 = 1+\frac{c^4x^2}{a^4z^2}$$

$$G = 1+z_y^2 = 1+\frac{c^4y^2}{a^4z^2} \quad ; \quad F = z_x\cdot z_y = \frac{c^4xy}{a^2b^2z^2}$$

$$dS = \sqrt{EG-F^2}\,dx\,dy = \sqrt{1+\frac{c^4x^2}{a^4z^2}+\frac{c^4y^2}{b^4z^2}+\cancel{\frac{c^8x^2y^2}{b^4a^4z^4}}-\cancel{\frac{c^8x^2y^2}{a^4b^4z^4}}}\,dx\,dy$$

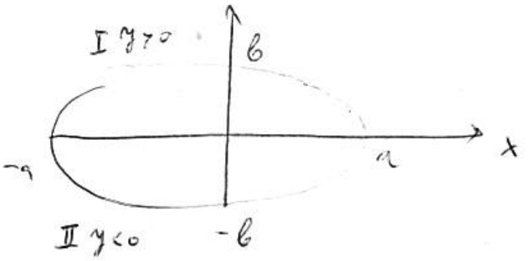
$$= \sqrt{1+z_x^2+z_y^2}\,dx\,dy$$

$$dS = \sqrt{\frac{1-\frac{x^2}{a^2}\left(1-\frac{c^2}{a^2}\right)-\frac{y^2}{b^2}\left(1-\frac{c^2}{b^2}\right)}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\,dx\,dy$$

$$2\sigma(x,y)\,dS(x,y) = \frac{Q}{2\pi abc}\,dx\,dy\int_{-A}^{A}\frac{1}{2\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\left[\underbrace{\delta\left(\frac{z}{c}-\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)}_{1}+\underbrace{\delta\left(\frac{z}{c}+\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)}_{1}\right]d\left(\frac{z}{c}\right)\cdot c$$

$$\boxed{\sigma(x,y)=\frac{Q}{4\pi ab}\,\frac{1}{\sqrt{1+\left(\frac{x}{a}\right)^2\left(\left(\frac{c}{a}\right)^2-1\right)+\left(\frac{y}{b}\right)^2\left(\left(\frac{c}{b}\right)^2-1\right)}}}$$

4. Елипса је равномерно наелектрисана густином наелектрисања λ по јединици дужине. Полуосе елипсе су a и b, а центар се налази у координатном почетку. Одредити запреминску густину наелектрисања елипсе у Декартовим координатама.



$$\rho=\sum_\alpha q_\alpha\,\delta^{(3)}(\vec r-\vec r_\alpha)=\int\lambda\,dl'\,\delta^{(3)}(\vec r-\vec r_\alpha')$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=\pm b\sqrt{1-\left(\frac{x}{a}\right)^2}\,;\; dy=\pm b\,\frac{\frac{x}{a^2}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx$$

$$dl=\sqrt{dx^2+dy^2}=dx\sqrt{1+\frac{b^2x^2}{a^4\left(1-\frac{x^2}{a^2}\right)}}$$

$$\rho=\int_{I}\lambda\,dl'\,\delta^{(3)}(\vec r-\vec r')+\int_{II}\lambda\,dl'\,\delta^{(3)}(\vec r-\vec r')$$

$$I:\ \vec r'=x'\vec e_x+b\sqrt{1-\left(\frac{x'}{a}\right)^2}\,\vec e_y\quad;\quad II:\ \vec r'=x'\vec e_x-b\sqrt{1-\left(\frac{x'}{a}\right)^2}\,\vec e_y$$

$$\rho(\vec r)=\lambda\int_{-a}^{a}\sqrt{1+\frac{b^2x'^2}{a^2(a^2-x'^2)}}\,dx'\,\delta(x-x')\,\delta\left(y-b\sqrt{1-\frac{x'^2}{a^2}}\right)\delta(z)$$

$$+\lambda\int_{-a}^{a}\sqrt{1+\frac{b^2x'^2}{a^2(a^2-x'^2)}}\,dx'\,\delta(x-x')\,\delta\left(y+b\sqrt{1-\frac{x'^2}{a^2}}\right)\delta(z)$$

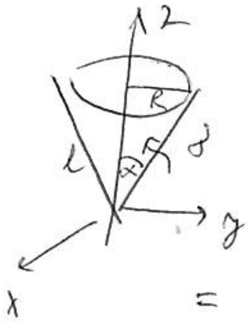
$$=\lambda\int_{-a}^{a}\sqrt{1+\frac{b^2x'^2}{a^2(a^2-x'^2)}}\,dx'\,\underline{\delta(x-x')}\,2b\sqrt{1-\frac{x'^2}{a^2}}\,\delta\left(y^2-b^2\left(1-\frac{x'^2}{a^2}\right)\right)\delta(z)$$

$$=\lambda\int_{-a}^{a}\sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}\,2b\sqrt{1-\frac{x^2}{a^2}}\,\delta\left(y^2-b^2\left(1-\frac{x^2}{a^2}\right)\right)\delta(z)\,\delta(x-x')$$

$$=\lambda\sqrt{\frac{a^4+(b^2-a^2)x^2}{a^2\cancel{(a^2-x^2)}}}\,2\cancel{b}\sqrt{\frac{\cancel{a^2-x^2}}{a^2}}\,\frac{1}{b^2}\,\delta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\delta(z)\,\eta(a^2-x^2)$$

$$=\boxed{\frac{2\lambda}{ab}\sqrt{a^2+\left(\left(\frac{b}{a}\right)^2-1\right)x^2}\;\delta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\delta(z)\,\eta(a^2-x^2)}$$

5. Конусна површ изводнице l и полупречника R, равномерно је наелектрисана површинском густином наелектрисања σ. Врх конуса је у координатном почетку, а висина је дуж z-осе. Наћи запреминску густину наелектрисања у сферним координатама и на основу тог резултата одредити диполни момент система.



$$\operatorname{tg}\alpha = \frac{R}{\sqrt{l^2-R^2}} \; ; \; \sin\alpha = \frac{R}{l}$$

$$\rho = \int \sigma \underbrace{dS'}\, \delta^{(3)}(\vec r - \vec r\,') = \int_0^l\int_0^{2\pi} \sigma\, \cancel{r'} dr' \cancel{\sin\alpha}\, d\varphi' \frac{1}{r^{\cancel{2}}\cancel{\sin\theta}}\, \delta(r-r')\delta(\theta-\alpha)\delta(\varphi-\varphi')$$

$$= \int_0^l \frac{\sigma}{r}\, dr'\, \delta(\theta-\alpha)\,\delta(r-r') = \boxed{\frac{\sigma}{r}\,\delta(\theta-\alpha)\,\eta(l-r)}, \quad \alpha = \arcsin\frac{R}{l}$$

dS на конусу $\quad \vec r = r\sin\alpha\cos\varphi\, \vec e_x + r\sin\alpha\sin\varphi\, \vec e_y + r\cos\alpha\, \vec e_z$

$u = r, \quad v = \varphi \qquad \dfrac{\partial \vec r}{\partial r} = \sin\alpha\cos\varphi\, \vec e_x + \sin\alpha\sin\varphi\, \vec e_y + \cos\alpha\, \vec e_z$

$$\frac{\partial \vec r}{\partial \varphi} = -r\sin\alpha\sin\varphi\, \vec e_x + r\sin\alpha\cos\varphi\, \vec e_y \; ;$$

$$E = \left(\frac{\partial \vec r}{\partial u}\right)^2 = \sin^2\alpha + \cos^2\alpha = 1 \qquad F = \left(\frac{\partial \vec r}{\partial u}\right)\left(\frac{\partial \vec r}{\partial v}\right) = -r\sin^2\alpha\sin\varphi\cos\varphi + r\sin^2\alpha\sin\varphi\cos\varphi = 0$$

$$G = \left(\frac{\partial \vec r}{\partial v}\right)^2 = r^2\sin^2\alpha$$

$$dS = \sqrt{EG - F^2}\, du\, dv = \boxed{r\sin\alpha\, dr\, d\varphi}$$

$$\vec p = \int \rho(\vec r)\, \vec r\, dV = \int_0^\infty\int_0^\pi\int_0^{2\pi} \frac{\sigma}{\cancel{r}}\,\delta(\theta-\alpha)\,\eta(l-r)\left(\cancel{r}\sin\theta\cancel{\cos\varphi}\, \vec e_x + \cancel{r}\sin\theta\cancel{\sin\varphi}\, \vec e_y + \cancel{r}\cos\theta\, \vec e_z\right) r^2\sin\theta\, dr\, d\theta\, d\varphi =$$

$$= \sigma\cdot 2\pi \int_0^\infty \eta(l-r)\cos\alpha\, r^2\sin\alpha\, \vec e_z\, dr = \sigma\, 2\pi\sin\alpha\cos\alpha\, \vec e_z \int_0^l r^2\, dr = \sigma\cdot 2\pi\sin\alpha\cos\alpha\, \frac{l^3}{3}\, \vec e_z$$

$$\vec p = \sigma\cdot 2\pi\, \frac{R}{l}\sqrt{1-\frac{R^2}{l^2}}\, \frac{l^3}{3}\, \vec e_z = \boxed{\frac{2\pi\sigma}{3}\, R\, l\sqrt{l^2-R^2}\, \vec e_z}$$

6. Запреминска густина наелектрисања је дата са $\rho(\vec r) = (\vec a\, \vec\nabla)\, \delta(\vec r)$, где је $\vec a = \mathrm{const}$. Наћи потенцијал и јачину поља у свакој тачки простора.

$$\varphi(\vec r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec r\,')}{|\vec r - \vec r\,'|}\, dV' = \frac{1}{4\pi\varepsilon_0}\int \frac{(\vec a\cdot\nabla')\,\delta(\vec r\,')}{|\vec r - \vec r\,'|}\, dV'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{a_i\left(\partial_i'\, \delta^{(3)}(\vec r\,')\right)}{|\vec r - \vec r\,'|}\, dV' = \frac{1}{4\pi\varepsilon_0}\int \partial_i'\left(\frac{a_i\,\delta^{(3)}(\vec r\,')}{|\vec r - \vec r\,'|}\right) dV' - \frac{1}{4\pi\varepsilon_0}\int \delta^{(3)}(\vec r\,')\, a_i\, \partial_i'\left(\frac{1}{|\vec r - \vec r\,'|}\right) dV'$$

јер $\delta^{(3)}(\vec r\,')\big|_{\pm\infty} = 0$

$$= -\frac{1}{4\pi\varepsilon_0}\int \delta^{(3)}(\vec r\,') \sum_i a_i\, \partial_i' \frac{1}{\sqrt{\sum_j (x_j - x_j')^2}} = -\frac{1}{4\pi\varepsilon_0}\int \delta^{(3)}(\vec r\,')\sum_i a_i \left(-\frac{1}{2}\right)\frac{1}{\sqrt{\sum_j (x_j-x_j')^2}^{\,3}}\cdot 2(x_i - x_i')(-1)$$

$$= -\frac{1}{4\pi\varepsilon_0}\,\frac{\sum a_i x_i}{|\vec r|^3} = \boxed{-\frac{1}{4\pi\varepsilon_0}\,\frac{\vec a\,\vec r}{r^3}}$$

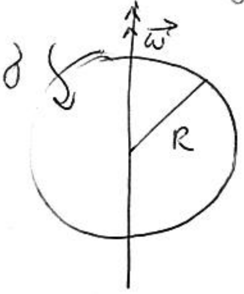
Потенцијал дипола $\vec p = -\vec a$, који се налази у коорд. почетку

$$\vec E = -\nabla\varphi = \frac{1}{4\pi\varepsilon_0}\,\partial_i\left(\frac{\vec a\vec r}{r^3}\right)\vec e_i = \frac{1}{4\pi\varepsilon_0}\,\partial_i\left(\frac{a_j x_j}{r^3}\right)\vec e_i$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{a_i}{r^3}\vec e_i + \frac{(\vec a\vec r)(-3)}{r^4}\cdot\partial_i\sqrt{\sum_j x_j^2}\;\vec e_i\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{\vec a}{r^3} - \frac{3(\vec a\vec r)}{r^4}\frac{1}{\cancel{2}r}\cancel{2}x_i\vec e_i\right)$$

$$= \boxed{\frac{1}{4\pi\varepsilon_0}\left(\frac{\vec a}{r^3} - \frac{3(\vec a\vec r)\,\vec r}{r^5}\right)}$$

7. Сфера полупречника R, равномерно је наелектрисана површинском густином наелектрисања σ и ротира константном угаоном брзином $\vec\omega$ око своје осе. Одредити запреминску густину струје $\vec j$ у простору, као и векторски потенцијал и магнетно поље у координатном почетку.



$$\rho = \int_S \sigma\, dS'\,\delta^{(3)}(\vec r - \vec r') = \int_0^{\pi}\int_0^{2\pi}\sigma\,\cancel{R^2\sin\theta'}\,d\theta'\,d\varphi'\,\frac{1}{\cancel{r^2\sin\theta}}\,\delta(r-R)\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')$$

$$\rho = \sigma\,\delta(r-R)$$

$$\vec j = \rho\vec v = \sigma\,\delta(r-R)\,\vec\omega\times\vec r = \sigma\,\delta(r-R)\,\omega\,\vec e_z\times(R\cos\theta\,\vec e_z + R\sin\theta\,\vec e_\rho)$$

$$\boxed{\vec j = \sigma R\omega\sin\theta\,\delta(r-R)\,\vec e_\varphi}$$

$$\vec A(\vec r) = \frac{\mu_0}{4\pi}\int\frac{\vec j(\vec r')\,dV'}{|\vec r-\vec r'|} = \frac{\mu_0}{4\pi}\int\frac{\sigma R\omega\sin\theta'\,\delta(r'-R)\,\vec e_{\varphi'}}{|\vec r-\vec r'|}\,dV'$$

$$\vec A(0) = \frac{\mu_0}{4\pi}\int\frac{\omega\sigma R\sin\theta'\,\delta(r'-R)\,\vec e_{\varphi'}}{\cancel{r'}}\,r'^2\sin\theta'\,dr'\,d\theta'\,d\varphi' = 0$$

$$\vec B(\vec r) = \frac{\mu_0}{4\pi}\int\frac{\vec j(\vec r')\times(\vec r-\vec r')}{|\vec r-\vec r'|^3}\,dV'$$

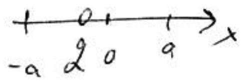
$$\vec B(0) = \frac{\mu_0}{4\pi}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi}\frac{\omega\sigma R\sin\theta'\,\delta(r'-R)\,\vec e_{\varphi'}\times(-\vec r')}{\cancel{r'^3}}\,\cancel{r'^2}\sin\theta'\,dr'\,d\theta'\,d\varphi'$$

$$= -\frac{\mu_0\sigma\omega}{4\pi}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi}\sin^2\theta\,\delta(r-R)\,\vec e_\varphi\times(R\cos\theta\,\vec e_z + R\sin\theta\,\vec e_\rho)\,dr\,d\theta\,d\varphi$$

$$= -\frac{\mu_0\sigma R\omega}{4\pi}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi}\sin^2\theta\,\delta(r-R)\,(\cos\theta\,\vec e_{\rho} - \sin\theta\,\vec e_z)\,dr\,d\theta\,d\varphi$$

$$= \frac{\mu_0\sigma R\omega}{4\pi}\vec e_z\,2\pi\int_0^{\pi}\sin^3\theta\,d\theta = \frac{\mu_0\sigma R\vec\omega}{2}\int_{-1}^{1}d\cos\theta\,(1-\cos^2\theta) = \frac{\mu_0\sigma R\vec\omega}{2}\left(2-\frac{2}{3}\right) = \boxed{\frac{2\mu_0\sigma R\vec\omega}{3}}$$

9. Наелектрисање q хармонијски осцилује дуж x-осе, $x = a\sin\omega t$, где је $a = const$. Одредити запреминску густину струје и проверити важење јне континуитета. Наћи средње вредности ρ и $\vec{j}$ за време једног периода и доказати да је $\int \langle\rho\rangle dV = q$.



$$\rho(\vec{r}) = \sum_\alpha q_\alpha \delta^{(3)}(\vec{r}-\vec{r}_\alpha) = q\,\delta^{(3)}(\vec{r}-a\sin\omega t\,\vec{e}_x)$$

$$\boxed{\rho(\vec{r}) = q\,\delta(x-a\sin\omega t)\,\delta(y)\,\delta(z)}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = a\omega\cos\omega t\,\vec{e}_x \qquad \boxed{\vec{j} = \rho(\vec{r})\vec{v} = a\omega q\cos\omega t\,\delta(x-a\sin\omega t)\,\delta(y)\,\delta(z)\,\vec{e}_x}$$

јна континуитета: $\quad \dfrac{\partial\rho}{\partial t} + \operatorname{div}\vec{j} = 0$

$$\frac{\partial\rho}{\partial t} = q\,\delta'(x-a\sin\omega t)\,\delta(y)\,\delta(z)\,(-a\omega)\cos\omega t \qquad {}'\text{– извод по } x-a\sin\omega t$$

$$\operatorname{div}\vec{j} = \frac{\partial j_x}{\partial x} = a\omega q\cos\omega t\,\delta'(x-a\sin\omega t)\,\delta(y)\,\delta(z)$$; Очигледно важи јна континуитета

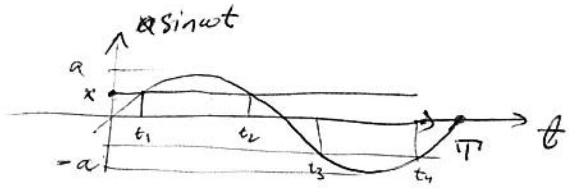
Период $T = \dfrac{2\pi}{\omega}$

$$\langle\rho\rangle \overset{def}{=} \frac{1}{T}\int_0^T \rho\,dt = \frac{1}{T}\int_0^T q\,\delta(x-a\sin\omega t)\,\delta(y)\,\delta(z)\,dt$$

$f(t) = x - a\sin\omega t \qquad \delta(f(t)) = \sum_{n=1}^{\infty} \dfrac{1}{|f'(t_n)|}\,\delta(t-t_n)$

$f'(t) = -a\omega\cos\omega t$

нуле фје $f(t)$ на интервалу $t \in [0,T]$



1° за $x>0 \quad f(t)=0$ за $t=t_1$ и $t=t_2 \qquad t_1 = \dfrac{1}{\omega}\arcsin\dfrac{x}{a}$; $t_2 = \dfrac{T}{2} - \dfrac{1}{\omega}\arcsin\dfrac{x}{a}$

2° за $x<0 \quad f(t)=0$ за $t=t_3$ и $t=t_4 \qquad t_3 = \dfrac{T}{2} + |t_1| \qquad t_4 = T - |t_1|$

$$1^\circ \quad x>0 \quad \langle\rho\rangle = \frac{1}{T}\int_0^T q\left(\frac{1}{|a\omega\cos\omega t_1|}\,\delta(t-t_1) + \frac{1}{|a\omega\cos\omega t_2|}\,\delta(t-t_2)\right)\delta(y)\,\delta(z)\,dt$$

$$|\cos\omega t_1| = |\cos\omega t_2| = |\cos\omega t_3| = |\cos\omega t_4| = \sqrt{1-\left(\frac{x}{a}\right)^2}$$

$$\langle\rho\rangle = \frac{1}{T}\int_0^T q\,\frac{1}{a\omega\sqrt{1-\left(\frac{x}{a}\right)^2}}\left(\delta(t-t_1)+\delta(t-t_2)\right)\delta(y)\,\delta(z)\,dt = \frac{2q}{a\omega 2\pi}\,\omega\,\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\,\delta(y)\,\delta(z)$$

$$\boxed{\langle\rho\rangle = \frac{q}{\pi}\,\frac{1}{\sqrt{a^2-x^2}}\,\delta(y)\,\delta(z)} \qquad \text{исто и за } 2^\circ.$$

$$\langle\vec{j}\rangle = \frac{1}{T}\int_0^T a\omega q\cos\omega t\,\delta(x-a\sin\omega t)\,\delta(y)\,\delta(z)\,\vec{e}_x\,dt$$

$$1^\circ \quad x>0 \quad \langle\vec{j}\rangle = \frac{1}{T}\int a\omega q\cos\omega t\,\frac{1}{|a\omega\cos\omega t_1|}\left(\delta(t-t_1)+\delta(t-t_2)\right)\delta(y)\,\delta(z)\,\vec{e}_x\,dt$$

$$= \frac{a\omega q}{T}\int \frac{1}{a\omega|\cos\omega t_1|}\left(\cos\omega t_1 + \cos\omega t_2\right)\delta(y)\,\delta(z)\,\vec{e}_x\,dt = 0$$

$$\cos\omega t_2 = -\cos\omega t_1$$

исто за 2°

$$\int \langle \rho \rangle_T \, dV = \iiint_{-\infty} \frac{q}{\pi} \frac{\delta(y)\delta(z)}{\sqrt{a^2-x^2}} \eta(a^2-x^2)\, dx\, dy\, dz$$

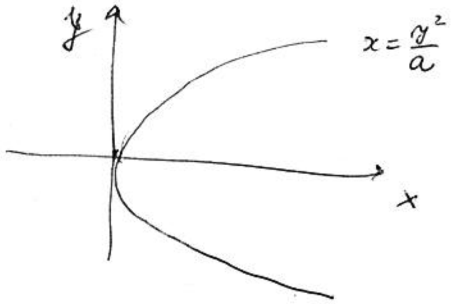
$$= \frac{q}{\pi}\int_{-a}^{a} \frac{dx}{\sqrt{a^2-x^2}} = \frac{q}{\pi}\int_{-a}^{a}\frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \frac{q}{\pi}\arcsin\left(\frac{x}{a}\right)\Big|_{-a}^{a} = \frac{q}{\pi}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = q$$

11. Парабола $y^2 = ax$, $z=0$ где је $a = const.$ наелектрисана је густином наелектрисања $\lambda(x) = q\sqrt{\frac{x}{4x+a}}$, $q = const$. Одредити:

(i) запреминску густину наелектрисања ρ

(ii) електрично поље на z-оси

(iii) електрични диполни момент у области у којој је $x < l$, при чему је $l > 0$ и $l = const$



$$dl' = \sqrt{dx'^2+dy'^2} = dy'\sqrt{1+\frac{4y'^2}{a^2}}$$

$$y = \sqrt{ax}, \quad x = \frac{y^2}{a}$$

$$(i) \quad \rho = \int_{-\infty}^{+\infty} \lambda(y')\, \underbrace{dl'}_{dy'\sqrt{\frac{4y'^2+a^2}{a^2}}}\, \delta(\vec r - \vec r')$$

$$= \int_{-\infty}^{+\infty} q\sqrt{\frac{y'^2/a}{\frac{4y'^2}{a}+a}}\, dy'\, \delta(x - \overset{f(y')}{\overset{\shortparallel}{x'}})\,\delta(y-y')\,\delta(z)\sqrt{\frac{4y'^2+a^2}{a^2}}$$

$$= q\int_{-\infty}^{+\infty}\sqrt{\frac{y'^2}{4y'^2+a^2}}\,\delta\left(x-\frac{y'^2}{a}\right)\delta(y-y')\,\delta(z)\, dy'\sqrt{\frac{4y'^2+a^2}{a^2}}$$

$$= \sqrt{\frac{4y^2+a^2}{a^2}}\, q\sqrt{\frac{y^2}{\cancel{4y^2+a^2}}}\,\delta\left(x-\frac{y^2}{a}\right)\delta(z)\int_{-\infty}^{+\infty}\delta(y-y')\,dy' = q\cancel{a}\sqrt{\frac{y^2}{\cancel{4y^2+a^2}}}\,\delta(y^2-ax)\,\delta(z)\sqrt{\frac{\cancel{4y^2+a^2}}{\cancel{a^2}}}$$

$$= q|y|\,\delta(y^2-ax)\,\delta(z) = \boxed{q\sqrt{ax}\,\delta(y^2-ax)\,\delta(z)}$$

$$(ii) \quad \vec E(\vec r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec r')(\vec r-\vec r')}{|\vec r-\vec r'|^3}\, d^3r'$$

$$\vec E(z\vec e_z) = \frac{1}{4\pi\varepsilon_0}\iiint_{-\infty}\frac{q\sqrt{ax'}\,\delta(y'^2-ax')\boxed{\delta(z')}\left(-x'\vec e_x - y'\vec e_y + (z-z')\vec e_z\right)}{\left(x'^2+y'^2+(z-z')^2\right)^{3/2}}\, dx'dy'\underline{dz'}$$

$$= \frac{1}{4\pi\varepsilon_0}\iint_{-\infty}^{+\infty}\frac{q\sqrt{ax'}\,\delta(y'^2-ax')(-x'\vec e_x - y'\vec e_y + z\vec e_z)}{(x'^2+y'^2+z^2)^{3/2}}\, dx'dy'$$

$$= \frac{\cancel{1}}{4\pi\varepsilon_0}\int_0^{+\infty}\int_{-\infty}^{+\infty} q\cancel{\sqrt{ax'}}\,\frac{1}{2\cancel{\sqrt{ax'}}}\left(\delta(y'-\sqrt{ax'})+\delta(y'+\sqrt{ax'})\right)\frac{(-x'\vec e_x - y'\vec e_y + z\vec e_z)}{(x'^2+y'^2+z^2)^{3/2}}\, dx'dy'$$

$$= \frac{q}{4\pi\varepsilon_0}\int_0^{\infty}\frac{-x'\vec e_x + z\vec e_z}{(x'^2+ax'+z^2)^{3/2}}\, dx' = \frac{q}{4\pi\varepsilon_0}\int_0^{\infty}\frac{-\frac{1}{2}d(x'^2+ax'+z^2)\,\vec e_x + \left(\frac{1}{2}a\vec e_x + z\vec e_z\right)dx'}{(x'^2+ax'+z^2)^{3/2}}$$

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\int_0 \frac{(\frac{1}{2}a\vec{e}_x + z\vec{e}_z)\,dx'}{((x'+a/2)^2+z^2-a^2/4)^{3/2}} + \frac{e_x}{(x'^2+ax'+z^2)^{1/2}}\Big|_0^\infty\right] \qquad \int\frac{dx}{(x^2\pm b^2)^{3/2}} = \frac{x}{\pm b^2\sqrt{x^2\pm b^2}} + \text{const.}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{(\frac{1}{2}a\vec{e}_x + z\vec{e}_z)(x'+a/2)}{(z^2-a^2/4)\sqrt{x'^2+ax'+z^2}} + \frac{\vec{e}_x}{(x'^2+ax'+z^2)^{1/2}}\right)\Big|_0^\infty$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{\frac{1}{2}a\vec{e}_x + z\vec{e}_z}{z^2-a^2/4}\left(1-\frac{a/2}{|z|}\right) - \frac{\vec{e}_x}{|z|}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\left(\left(1-\frac{a}{2|z|}\right)\frac{a}{2(z^2-\frac{a^2}{4})} - \frac{1}{|z|}\right)\vec{e}_x + \left(1-\frac{a}{2|z|}\right)\frac{z}{z^2-a^2/4}\,\vec{e}_z\right]$$

(iii) $$\vec{p} = \int \rho\vec{r}\,dV = \int_0^\ell\int_{-\infty}^{+\infty}\int \varrho\sqrt{ax}\,\delta(y^2-ax)\,\delta(z)\,(x\vec{e}_x + y\vec{e}_y + \cancel{z\vec{e}_z}^{\,0})\,dx\,dy\,dz$$

$$= \int_0^\ell\int_{-\infty}^{+\infty} \varrho\cancel{\sqrt{ax}}\,\frac{1}{2\cancel{\sqrt{ax}}}\left(\delta(y-\sqrt{ax}) + \delta(y+\sqrt{ax})\right)(x\vec{e}_x + y\vec{e}_y)\,dx\,dy$$

$$= \frac{\varrho}{\cancel{2}}\int_0^\ell(\cancel{2}x\vec{e}_x + \cancel{\sqrt{ax}\,\vec{e}_y} - \cancel{\sqrt{ax}\,\vec{e}_y})\,dx = \boxed{\varrho\frac{\ell^2}{2}\vec{e}_x}$$